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Convergence of cubic piecewise function

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Abstract

In this paper we construct a piecewise cubic polynomial function for approximation of a class of functions. From application point of view an explicit representation of the function is obtained. The convergence of constructed function has been discussed. The proofs of the results are simpler and shorter.

MSC: 41A15

Keywords: Polynomial splines; Convergence; Explicit representation

1. Definitions and notations:

Let Δ be a mesh of $[0, 1]$ given by $0=x_0 < x_1 < x_2 < \cdots < x_n=1$ such that $x_i - x_{i-1} = h_i$, $i=1, 2, \ldots, n$, and let π_m be the set of all real algebraic polynomials of degree at most m. The class of deficient polynomial splines of degree m with deficiency k, $k < m-1$ is defined as $S(m_k, \Delta) = \{s(x) : s(x) \in C^{m-k-1}[0, 1],\ s(x) \in \pi_m,\ x \in [x_{i-1}, x_i],\ i = 1, 2, \ldots, n\}$.

For the sake of convenience, we write $f(x_i) = f_i$; and the matrix

$$\begin{pmatrix} q & p & 0 & 0 & \ldots & 0 & c_1 \\ r & q & p & 0 & \ldots & 0 & 0 \\ 0 & r & q & p & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ c_2 & 0 & 0 & 0 & \ldots & r & q \end{pmatrix}$$

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is denoted by $C_n(r, q, p; c_1, c_2)$. We write $|C_n(r, q, p; c_1, c_2)|$ for the determinant of the matrix; when $c_1 = c_2 = 0$ it is represented by $|C_n|$.

We also write $\lambda = \alpha(m_2 - m_1) - 2m_2 - m_1 + 1$, $\underline{h} = \max_i h_i$, and $\|f\| = \sup_x |f(x)|$.

2. Representation by spline functions

In the literature, the approximation by means of spline interpolation has been studied extensively. There are interesting results of Schoenberg [10], de Boor [2], Meir and Sharma [7], Dikshit [3], Dikshit and Powar [4], Kumar and Govil [6], Rana [8], Rana and Purohit [9] etc., dealing with cubic spline interpolation. It was observed in [7] that there is no priori reason to obtain approximating spline just by interpolating at one point of subinterval. In order to interpolate at two points, one needs one more degree of freedom and that was obtained by reducing the continuity requirement of derivative of second degree of the spline functions. Again a natural question is that—can we take the condition of the continuity of the second derivative of spline in some other form? For example, on a nature of the spline function itself rather than interpolation at one more point. Further, in the former approach we need lesser number of data than that required for deficient spline function used for approximation of a function. In view of this, we dispense with the requirement of second derivative continuity by the condition

$$\begin{aligned}&\alpha s(x_{i-1} + m_1 h_i) + (1-\alpha) s(x_{i-1} + m_2 h_i)\\ &\quad = s(x_{i-1} + h_i(\alpha m_1 + (1-\alpha) m_2)), \quad i = 1, 2, \ldots, n,\end{aligned} \tag{2.1}$$

where $0 < m_1 < m_2 < 1$ and $0 < \alpha < 1$. We denote such class of spline function by $S(3_1^c, \Delta)$.

The error estimation reflects effect of the parameter used in (2.1). Also the closeness of $f'(x)$ and $s'(x)$ can be obtained by the selection of parameters α, m_1, m_2 in (2.1). We also obtain explicit presentation of the spline functions. This presentation is useful from the application point of view.

First we establish the following:

Theorem 1. *There exist a unique* 1*-periodic spline function* $s \in S(3_1^c, \Delta)$ *satisfying interpolatory condition*

$$f(x_i) = s(x_i), \quad i = 1, 2, \ldots, n,$$

if and only if $\lambda \neq -\frac{1}{2}$.

It may be noticed that $-2 < \lambda < 2$.

3. Proof of Theorem 1

We write $s'(x_i) = M_i$, $i = 0, 1, \ldots, n$. It is direct to see that cubic polynomial in $I_i = [x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$, can be presented as

$$\begin{aligned}h_i^2 s(x) = &-M_{i-1}\left\{\frac{(x_i - x)^3}{3} + \frac{(x - x_{i-1})^3}{3} + (x - x_{i-1})^2 (x_i - x)\right\}\\ &- M_i (x - x_{i-1})^2 (x_i - x) + \eta_i (x - x_{i-1})^2\\ &\times \{3(x_i - x) + (x - x_{i-1})\} + \delta_i h_i^2,\end{aligned} \tag{3.1}$$

where η_i and δ_i are constants. The point matching condition yields

$$f(x_i) = -\frac{h_i}{3} M_{i-1} + \eta_i h_i + \delta_i, \quad i = 1, 2, \ldots, n. \tag{3.2}$$

From (3.1) we get

$$s(x_{i-1} + m_1 h_i) = M_{i-1} h_i (m_1(1-m_1)^2 - \tfrac{1}{3}) - M_i h_i m_1^2 (1 - m_1) + \eta_i h_i m_1^2 (3 - 2m_1) + \delta_i.$$

Taking in the above relation m_1 as m_2, and $\alpha m_1 + (1-\alpha) m_2$ we get the terms of Eq. (2.1). Thus, condition (2.1), gives

$$\lambda M_i + (\lambda + 1) M_{i-1} = \eta_i (2\lambda + 1), \quad i = 1, 2, \ldots, n. \tag{3.3}$$

From the continuity condition of $s(x)$ at x_i, $i = 1, 2, \ldots, n-1$, we get

$$\tfrac{1}{3}[h_{i+1} M_i - h_i M_{i-1}] + \eta_i h_i = \delta_{i+1} - \delta_i. \tag{3.4}$$

Observing that for the periodic f, $M_{n+r} = M_r$, $r = 0, 1$. By the continuity requirement of $s(x)$ we get that values of $s(1)$ and $s(0)$, obtained from the splines in $[x_{n-1}, x_n]$ and $[x_0, x_1]$, respectively, are equal. This gives that the continuity condition (3.4) holds for $i = n$ also, if we write $\delta_{n+1} = \delta_1$ and $h_{n+1} = h_1$. We eliminate the constants η_i and $\delta_i' s$ with the help of (3.2)–(3.4), and obtain that for $i = 1, 2, \ldots, n$,

$$U_i = \lambda M_{i+1} + (\lambda + 1) M_i, \tag{3.5}$$

where $U_i = (h_{i+1})^{-1} (f_{i+1} - f_i)(2\lambda + 1)$.

The spline function exists uniquely if and only if the system of Eq. (3.5) has unique solutions for $M_i' s$. It is direct to see that the coefficient matrix is $C_n(0, \lambda+1, \lambda; 0, \lambda)$, and it is diagonally dominant if $\lambda \neq -\frac{1}{2}$. For $\lambda = -\frac{1}{2}$, the determinant is zero. This establishes the theorem.

Remark. For the non-periodic splines, Eqs. (3.5) hold for $i = 1, 2, \ldots, n-1$. For determining $(n+1)$ unknowns $M_0, M_1, \ldots, M_n$ two additional condition must be imposed, known as the end conditions. One of the possible way is assuming $M_0 = M_n = 0$. In this case the determinant of the matrix is $(\lambda+1)^{n-1}$. So the solution does not exist if and only if $\lambda = -1$. For the purpose of error estimation, the norm of the inverse matrix can be obtained by Lemma (c) given in the next section.

4. Explicit representation

We make use of the following result [5, Lemma 1, p. 72], also [6, p. 177].

Lemma. (a) We have,

(i) $|C_n| = (\beta_1^{n+1} - \beta_2^{n+1})/(\beta_1 - \beta_2)$ where $\beta_1 + \beta_2 = q$, and $\beta_1 - \beta_2 = \sqrt{q^2 - 4pr}$.
(ii) $|C_n(r, q, p; r, p)| = q|C_{n-1}| - 2pr|C_{n-2}| + (-1)^{n+1}(p^n + r^n)$.

(b) If $C_n(r, q, p; r, p)$ be a non-singular matrix, then the inverse $(\widehat{a}_{ij})$ is given by

$$\widehat{a}_{ij} = \begin{cases} \dfrac{(-1)^{i-j}\{r^{i-j}|C_{n-(i-j)-1}| + (-1)^n p^{n-(i-j)}|C_{i-j-1}|\}}{|C_n(r, q, p; r, p)|}, & j < i, \\ \dfrac{(-1)^{j-i}\{p^{j-i}|C_{n-(j-i)-1}| + (-1)^n r^{n-(j-i)}|C_{j-i-1}|\}}{|C_n(r, q, p; r, p)|}, & j > i, \\ \dfrac{|C_{n-1}|}{|C_n(r, q, p; r, p)|}, & j = i, \end{cases}$$

where $|C_{-1}| = 0$, $|C_0| = 1$.

(c) The inverse of $C_n(r, q, p; 0, 0)$, is the matrix $(\widehat{b_{ij}})$, where

$$\widehat{b}_{ij} = \begin{cases} \dfrac{(-r)^{i-j}|C_{j-1}||C_{n-i}|}{|C_n|}, & i \geqslant j, \\ \dfrac{(-p)^{j-i}|C_{i-1}||C_{n-j}|}{|C_n|}, & i \leqslant j. \end{cases}$$

Now we proceed to obtain explicit representation for $s(x)$. Obviously, $|C_n(0, \lambda+1, \lambda; 0, 0)| = (\lambda+1)^n$. Consequently, by the Lemma (a)—(ii) $|C_n(0, \lambda + 1, \lambda; 0, \lambda)| = (\lambda + 1)^n + (-1)^{n+1}\lambda^n$. Since λ assumes only finite real values, the matrix is non-singular if and only if $\lambda \neq -\frac{1}{2}$. Further, by Lemma (b) we have, $C_n^{-1}(0, \lambda + 1, \lambda; 0, \lambda) = (\widehat{y}_{ij})$, where

$$\widehat{y}_{ij} = \begin{cases} \dfrac{|\lambda|^{n-i+j}|\lambda + 1|^{i-j-1}}{|C_n(0, \lambda + 1, \lambda; 0, \lambda)|}, & j < i, \\ \dfrac{|\lambda|^{j-i}|\lambda + 1|^{n-j+i-1}}{|C_n(0, \lambda + 1, \lambda; 0, \lambda)|}; & j \geqslant i; \end{cases}$$

here $0^{i-j} = 1$ even for $i = j$. This enables to get $M_i's$, that is, $(M_i) = (\widehat{y}_{ij})(U_j)$. On substituting values for $M_i's$, on the following equation obtained from (3.1) on elimination of η_i and δ_i by (3.2) and (3.3), we get explicit form of $s(x)$.

$$\begin{aligned} h_i^2 s(x) = M_{i-1} & \left\{ -\frac{(x_i - x)^3}{3} + \frac{\lambda + 2}{2\lambda + 1} \left[\frac{(x - x_{i-1})^3}{3} + (x - x_{i-1})^2(x_i - x) - \frac{h_i^3}{3} \right] \right\} \\ & + \frac{M_i}{2\lambda + 1}\{\lambda(x - x_{i-1})^3 + (\lambda - 1)(x - x_{i-1})^2(x_i - x) - \lambda h_i^3\} + f_i h_i^2. \end{aligned} \tag{4.1}$$

We consider two special cases from the application point of view. For $\lambda = -1$ or 0 that is when $m_1 = (m_2(\alpha - 2) + 2)/(\alpha + 1)$ or $m_1 = (m_2(\alpha - 2) + 1)/(\alpha + 1)$, we get $\widehat{y}_{ij} = 1$ for $i = j$ otherwise 0. For this case $M_i = U_i$.

5. Rate of convergence

We now consider error of approximation of the spline function obtained in Theorem 1. Let $e(x) = s(x) - f(x)$, and let $\widehat{A}(\lambda) = \|C_n^{-1}(0, \lambda + 1, \lambda; 0, \lambda)\|$ be row-max-norm of the inverse of the coefficient matrix of $M_i's$ of the system of equations (3.5). It can be seen from Lemma (b) that

$$\widehat{A}(\lambda)=\frac{|\lambda|^n-|\lambda+1|^n}{|(\lambda+1)^n+(-1)^{n+1}\lambda^n|(|\lambda|-|\lambda+1|)}. \tag{5.1}$$

From (3.5) we get

$$\underline{e}'=\max_i |e_i'|\leqslant \widehat{A}(\lambda)\|(U_i)-C_n(0,\lambda+1,\lambda;0,\lambda)(f_i')\|.$$

That is

$$\underline{e}'\leqslant \widehat{A}(\lambda)\max_i |[f(x_{i+1})-f(x_i)](2\lambda+1)h_{i+1}^{-1}-\lambda f_{i+1}'-(\lambda+1)f_i'|.$$

Using suitably Taylor's expansion, viz.

$$f(x_{i+1})-f(x_i)=h_{i+1}f'(\eta_{i+1}),\quad x_i\leqslant\eta_{i+1}\leqslant x_{i+1}.$$

As usual we denote the modulus of continuity of a function by ω. On simplification we get

$$\underline{e}'\leqslant \widehat{A}(\lambda)(|\lambda+1|+|\lambda|)\omega(f',\underline{h}). \tag{5.2}$$

From (4.1) we get

$$\begin{aligned}s'(x)={}&M_{i-1}\frac{1}{h_i^2}\left[(x_i-x)^2+\frac{2\lambda+4}{2\lambda+1}(x-x_{i-1})(x_i-x)\right]\\&+M_i\frac{1}{h_i^2}\left[(x-x_{i-1})^2+\frac{2\lambda-2}{2\lambda+1}(x-x_{i-1})(x_i-x)\right].\end{aligned}$$

We have for $x\in[x_{i-1},x_i]$,

$$\begin{aligned}e'(x)={}&\frac{1}{h_i^2}\left[(x_i-x)^2+\frac{2\lambda+4}{2\lambda+1}(x-x_{i-1})(x_i-x)\right][M_{i-1}-f_{i-1}']\\&+\frac{1}{h_i^2}\left[(x-x_{i-1})^2+\frac{2\lambda-2}{2\lambda+1}(x-x_{i-1})(x_i-x)\right][M_i-f_i']\\&+\frac{1}{h_i^2}\left[(x_i-x)^2+\frac{2\lambda+4}{2\lambda+1}(x-x_{i-1})(x_i-x)\right][f_{i-1}'-f'(x)]\\&+\frac{1}{h_i^2}\left[(x-x_{i-1})^2+\frac{2\lambda-2}{2\lambda+1}(x-x_{i-1})(x_i-x)\right][f_i'-f'(x)].\end{aligned}$$

Writing $x=x_{i-1}+\theta h_i,\ 0\leqslant\theta\leqslant 1$, we see that

$$\begin{aligned}e'(x)={}&\frac{1}{|2\lambda+1|}\{(1-\theta)[2\lambda+3\theta+1]e_{i-1}'+\theta[2\lambda+3\theta-2]e_i'\}\\&+\frac{1}{|2\lambda+1|}\{(1-\theta)[2\lambda+3\theta+1][f_{i-1}'-f'(x)]\\&+\theta[2\lambda+3\theta-2][f_i'-f'(x)]\}.\end{aligned}$$

We denote the expression under the curly brackets by Σ_1 and Σ_2, respectively. We have

$$
\begin{aligned}
|\Sigma_2| &= \frac{1}{|2\lambda+1|} |[3\theta^2 + 2\theta(\lambda-1)][f_i' - f_{i-1}'] + (2\lambda+1)[f_{i-1}' - f'(x)]| \\
&\leqslant \left[1 + \left|\frac{3\theta^2 + 2\theta(\lambda-1)}{2\lambda+1}\right|\right] \omega(f', \underline{h}) \\
&= \left[1 + \frac{1}{3|2\lambda+1|} \max\{(1-\lambda)^2, 3|2\lambda+1|\}\right] \omega(f', \underline{h}).
\end{aligned}
$$

Further, for the case $-1 \leqslant 2\lambda + 3\theta < 2$, we have

$$
\begin{aligned}
|\Sigma_1| &= \frac{1}{|2\lambda+1|} [(1-\theta)[2\lambda+3\theta+1]e_{i-1}' - \theta[2\lambda+3\theta-2]e_i'] \\
&\leqslant \frac{1}{|2\lambda+1|} [-6\theta^2 - 4\theta(\lambda-1) + 2\lambda + 1]\underline{e}' \\
&\leqslant \frac{2\lambda^2 + 2\lambda + 5}{3|2\lambda+1|} \underline{e}'.
\end{aligned}
$$

It can be seen that for $2\lambda + 3\theta \geqslant 2$ and $2\lambda + 3\theta < -1$, $|\Sigma_1| \leqslant \underline{e}'$.

Since

$$
\max\left\{\frac{2\lambda^2 + 2\lambda + 5}{3|2\lambda+1|}, 1\right\} = \frac{2\lambda^2 + 2\lambda + 5}{3|2\lambda+1|},
$$

from (5.2) we get $|e'(x)| \leqslant K(\lambda)\omega(f', \underline{h})$, where

$$
\begin{aligned}
K(\lambda) = {} & \frac{2\lambda^2 + 2\lambda + 5}{3|2\lambda+1|} \widehat{A}(\lambda)(|\lambda+1| + |\lambda|) \\
& + \left\{1 + \frac{1}{3|2\lambda+1|} \max\{(1-\lambda)^2, 3|2\lambda+1|\}\right\}.
\end{aligned}
\tag{5.3}
$$

From the hypothesis $e_i = 0$, we get $|e(x)| \leqslant \int_{x_{i-1}}^{x} |e'(x)|\,\mathrm{d}x$ for $x \in (x_{i-1}, x_i)$, $i = 1, 2, \ldots, n$. Combining all these including (5.2) we get the following:

Theorem 2. *The spline function of Theorem* 1 *approximates* $f \in C^1[0,1]$ *with the error as given below*

$$
\begin{aligned}
&\max_i |e_i'| \leqslant \widehat{A}(\lambda)(|\lambda+1| + |\lambda|)\omega(f', \underline{h}), \\
&|e'(x)| \leqslant K(\lambda)\ \omega(f', \underline{h}), \\
&|e(x)| \leqslant K(\lambda)\underline{h}\omega(f', \underline{h}),
\end{aligned}
$$

where $\widehat{A}(\lambda)$ *and* $K(\lambda)$ *are defined in* (5.1) *and* (5.3).

It is interesting to observe that the constant occurring in the above can be made smaller by choosing λ suitably where $\lambda = \alpha(m_2 - m_1) - 2m_2 - m_1 + 1$. An interesting case is when $\lambda = -1$ or 0. It is direct to see that $K(-1)=4$ and $K(0)=\frac{11}{3}$. This shows that this error expression is better than the corresponding approximation [1, p. 27]. Besides this the explicit representation comes immediately.

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